An anisotropic model of kinetic roughening: the strong-coupling regime

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We study the *strong coupling* (SC) *limit* of the anisotropic Kardar-Parisi-Zhang (KPZ) model. A systematic mapping of the continuum model to its lattice equivalent shows that in the SC limit, anisotropic perturbations destroy all spatial correlations but retain a temporal scaling which shows a remarkable cross-over along one of the two spatial directions, the choice of direction depending on the relative strength of anisotropicity. The results agree with exact numerics and is expected to settle the long-standing SC problem of a KPZ model in the infinite range limit.

A topic of much interest in the field of non-equilibrium statistical physics in the past few years has been the ubiquitous Kardar-Parisi-Zhang (KPZ) model [1–3]. Although many of the theoretical issues concerning the *weak coupling* (WC) *regime* of the model, the regime amenable to perturbation theories, have generally been well studied [2–7], the same can not be said about the *strong coupling* (SC) *regime*. Barring occasional studies [8–13], this regime has largely remained unexplored mainly due to a lack of theoretical tools in dealing with such a non-perturbative system, as well as due to the inherent complex character of the problem. Perturbative field theories [4, 9, 10] could probe the system close to the SC-WC phase boundary but not the SC regime itself.

Three open problems concerning the SC regime are well known: 1) what is the nature of the universality class in this regime, that is if one exists? 2) what is the value of the upper critical dimension d_c beyond which the dynamic exponent $z = 2$ and the roughness exponent $\chi = 0$? 3) what is the effect of anisotropic perturbations on the SC regime as well as on d_c? Starting with the latter, the role of anisotropy is a well-studied problem but mostly in the WC regime and that too with contradictory outcomes, eg. Wolf [14] claims the triviality of such fluctuations whereas Täuber-Frey [9], Tang-Kardar-Dhar [15], Mukherji-Bhattacharjee [16] and Hwa [17] claim quite the opposite. The issue of the upper critical dimension has also remained a highly debated issue all along. Claims toward its existence [11, 19–22] have been refuted by equally powerful arbitrations toward the opposite [9, 10, 13, 18]. The question concerning universality class of KPZ-type models in the SC regime has remained largely unexplored though (exceptions are [11, 13, 18]).

The model we study in tackling all three contentious issues together is the anisotropic KPZ model [2] and the theoretical approach we rely upon is a generalization of the "infinite range mean-field" (IRMF) technique popularized by Marsili and Bray [8] for a standard KPZ model. To achieve such goals, we use a direct mapping of the con-tinuum model to its discretized lattice equivalent [23] and study the resultant Fokker-Planck type master equation thereof [8]. This has the advantage of a non-perturbative approach to the problem which could be profitably used to study the SC phase. The analytical results are later complemented by a direct numerical simulation of the spatially anisotropic equations of motion. With respect to the spatial directions x and y, the anisotropic KPZ model can be represented as

$$\frac{\partial h}{\partial t}(x,y,t) = \nu \sum_i^{x,y} \partial_i^2 h + \sum_i^{x,y} \lambda_i (\partial_i h)^2 + \eta(x,y,t) \quad (1)$$

where terms have their usual meanings and the noise $\eta(x,y,t)$ is white with strength D. Our interest is in the $d > d_c$, $\lambda_i > \lambda_i^{\text{critical}}$ regime $(i = x, y)$. Resorting to IRMF, we now map the continuum model defined in eqn (1) to a square lattice model considering only nearest-neighbor (nn) interactions. For a 2+1 dimensional Euclidean space, this means the number N of the nn sites is equal to 4 (=2d). If $h_t^{\alpha,\beta} \equiv h(x,y,t)$ represents the height at the lattice point (α,β), an ensemble averaging along x (*i. e.* along α) gives $< h_t(\alpha) >_x = \frac{1}{N}\sum_{\alpha=1}^N h^{\alpha,\beta}(t)$ while that along y (*i. e.* along β) gives $< h_t(\beta) >_y = \frac{1}{N}\sum_{\beta=1}^N h^{\alpha,\beta}(t)$. We now introduce the local height fluctuation operators respectively along x and y at (α,β): $\psi_x(t) = h^{\alpha,\beta}(t) - < h_t(\alpha) >_x$ and $\psi_y(t) = h^{\alpha,\beta}(t) - < h_t(\beta) >_y$. Using a Taylor expansion up to the discretized second derivative, the h-description can now be mapped over to the ψ_i-description: $\partial_x^2 h_t(\alpha) = -\psi_x(t)$, $\partial_y^2 h_t(\beta) = -\psi_y(t)$ and $2|\partial_x h_t|^2 = < \psi_x^2(\alpha) >_t + \psi_x^2$, $2|\partial_y h_t|^2 = < \psi_y^2(\beta) >_t + \psi_y^2$ [8]. Further symmetry properties of (ψ_x, ψ_y) eg. $< \psi_x(\alpha) >_x = < \psi_y(\beta) >_y = 0$ & $< \psi_x(\beta) >_y = - < \psi_y(\alpha) >_x$, etc assume non-trivial significance in defining the structure of the potential function Φ (eqn 4). As already shown in [8, 24], non-linear equations of motion of the type do not have a stationary state: $\psi_i \to \infty$ $(i = x,y)$ as $t \to \infty$. To avoid such inconsistencies, one can resort to a regularization

scheme involving the introduction of dimensionally "irrelevant operators" in the free energy of the model as in [8], thereby generating a modified equation of motion

$$\frac{\partial h}{\partial t} = \nu \sum_{i=x,y} \partial_i{}^2 h + \sum_{i=x,y} \lambda_i \ (\partial_i h)^2$$
$$+ \sum_{i=x,y} \kappa_i \ (\partial_i{}^2 h)(\partial_i h)^2 + \eta(x,y,t) \qquad (2)$$

It is easy to see that the surface tension and "irrelevant" terms together contribute in renormalizing ν, effectively amounting to a smoothening of the growing surfaces. In the infinite-range limit, this allows the $h \leftrightarrow \psi$ mapping to define a coupled set in (ψ_x, ψ_y):

$$\partial_t \psi_i = \Gamma_i [\mu(t) - \Phi(\psi_x, \psi_y, \xi)] + \eta_i \ (i = x,y) \qquad (3)$$

where $\Gamma_x = 1$ and $\Gamma_y = \frac{1}{\xi}$. The white noises (η_x, η_y) are related to η through a renormalized noise strength $DN^{-1/2}$ while the potential

$$\Phi(\psi_x, \psi_y, \xi) = \sum_i^{x,y} [1+s_i{}^t]\psi_i - \sum_i^{x,y} \zeta_i g_i \psi_i{}^2 + \sum_i^{x,y} \zeta_i \psi_i{}^3 \qquad (4)$$

and the system parameters are as follows

$$\zeta_x = 1, \zeta_y = \xi, \mu(t) = \ <\Phi>_{x,y} \qquad (5a)$$
$$s_{xy}{}^t = \ <\psi_x \psi_y>_{x,y}, s_i{}^t = \ <\psi_i{}^2>_i \qquad (5b)$$

$(\xi = \sqrt{\frac{\kappa_x}{\kappa_y}}, \ g_x = \frac{\lambda_x}{\sqrt{2\nu\kappa_x}} \ \& \ g_y = \frac{\lambda_y}{\sqrt{2\nu\kappa_y}})$. Using a common potential V_{tot},

$$V_{\text{tot}} = \frac{1}{1+\xi}\{-\mu \sum_i^{x,y} \psi_i + \frac{1}{2} \sum_i^{x,y} \zeta_i (1+s_i)\psi_i{}^2$$
$$+ \ [(1+\xi) + \sum_{i \neq j}^{x,y} \zeta_i s_i]\psi_i \psi_j - \frac{1}{3}\sum_i^{x,y} \zeta_i g_i \psi_i{}^3$$
$$- \sum_{i \neq j}^{x,y} \zeta_i g_i \psi_i{}^2 \psi_j + \frac{1}{4}\sum_i^{x,y} \zeta_i \psi_i{}^4 + \sum_{i \neq j}^{x,y} \psi_i{}^3 \psi_j\} \qquad (6)$$

the coupled eqns in (3) can now be represented as a Fokker-Planck set:

$$\partial_t \psi_i = \Gamma_i \frac{\partial V_{\text{tot}}}{\partial \psi_i} + \eta_i \ (i = x,y) \qquad (7)$$

subject to the condition

$$\xi^2[(1+s_y) - 2g_y \psi_y + 3\psi_y{}^2] = (1+s_x) - 2g_x \psi_x + 3\psi_x{}^2 \qquad (8)$$

$s_x{}^\infty$, $s_y{}^\infty$ and μ^∞ can be evaluated by studying the stability properties of V_{tot} using the Hessian $[H = \begin{pmatrix} \frac{\partial^2 V_{\text{tot}}}{\partial \psi_x{}^2} & \frac{\partial^2 V_{\text{tot}}}{\partial \psi_x \partial \psi_y} \\ \frac{\partial^2 V_{\text{tot}}}{\partial \psi_y \partial \psi_x} & \frac{\partial^2 V_{\text{tot}}}{\partial \psi_y{}^2} \end{pmatrix}]$ description, in the *low noise* limit. The resultant analyzes suggest optima for the system at $(\psi_x{}^\pm, \psi_y{}^\pm)$ where $\psi_i{}^\pm = (\frac{g_i \pm \sqrt{g_i{}^2 - 4(1+s_i)}}{2} \ (i = x,y)$. $\psi_i{}^\pm$ represent minima if $\xi(= \frac{g_x}{g_y}) > 1$ and vice versa. As we will shortly see, this leads to a cross-over along y (for $\xi > 1$) or along x ($\xi < 1$), a trait not to be found in the isotropic model.

In the limit $D \to 0^+$, the equilibrium potential is given by $(1+\xi)V_\infty(\psi_x,\psi_y;\xi) = \tilde{V} + \sum_{i,j}^{x,y} \frac{\zeta_i}{4}(\psi_i - \psi_i{}^-)^2(\psi_j - \psi_j{}^+)^2 + \frac{1}{2}\Pi_{i \neq j}^{x,y}(\psi_i - \psi_i{}^-)(\psi_j - \psi_j{}^+)$ leading to the stationary state probability distribution function $P_\infty(\psi_x, \psi_y, \xi) = \frac{\alpha}{2}\delta(\psi_x - \psi_x{}^-)\delta(\psi_y - \psi_y{}^-) + \frac{(1-\alpha)}{2}\delta(\psi_x - \psi_x{}^+)\delta(\psi_y - \psi_y{}^+) + \frac{\beta}{2}\delta(\psi_x - \psi_x{}^-)\delta(\psi_y - \psi_y{}^+) + \frac{(1-\beta)}{2}\delta(\psi_x - \psi_x{}^+)\delta(\psi_y - \psi_y{}^-)$, where α, β are probabilities of transitions between "pure" $(\phi_x{}^\pm, \phi_y{}^\pm)$ and "mixed" $(\phi_x{}^\pm, \phi_y{}^\mp)$ states respectively. Together with identities like $<\psi_i>_\infty = 0$, this gives us $s_i{}^\infty = -\psi_i{}^+\psi_i{}^-$ (i=x,y) and $\mu^\infty = -\frac{g_x}{6}s_x{}^\infty - \xi\frac{g_y}{6}s_y{}^\infty$. The key signature of anisotropicity is, however, encapsulated in the non-zero value of the cross-coupling correlator s_{xy}. It ensures that fluctuations along x *can influence* those along y, thereby leading to $s_{xy}{}^\infty = -\frac{1}{4}(\psi_x{}^+ + \psi_x{}^-)(\psi_y{}^+ + \psi_y{}^-)$, or as in the *low noise* limit: $s_{xy}{}^\infty(g_x, g_y, D \to 0^+) = -\frac{1}{9}g_x g_y$.

We now address the question of spatio-temporal scaling (or the lack of it) by studying the two-point structure function $<(h - <h>)>^2 = <\psi_i{}^2> \ (i = x,y)$ away from the stationary state. For the temporal probability distribution, we propose a structure similar to $P_\infty(\psi_x, \psi_y; \xi)$, though now with time dependent transition probabilities (α_t, β_t) defined against time-dependent states $(\psi_{x,t}, \psi_{y,t})$: $P_t(\psi_{x,t}, \psi_{y,t}, \xi) = \frac{\alpha_t}{2}\delta(\psi_{x,t} - \psi_{x,t}{}^-)\delta(\psi_{y,t} - \psi_{y,t}{}^-) + \frac{(1-\alpha_t)}{2}\delta(\psi_{x,t} - \psi_{x,t}{}^+)\delta(\psi_{y,t} - \psi_{y,t}{}^+) + \frac{\beta_t}{2}\delta(\psi_{x,t} - \psi_{x,t}{}^-)\delta(\psi_{y,t} - \psi_{y,t}{}^+) + \frac{(1-\beta_t)}{2}\delta(\psi_{x,t} - \psi_{x,t}{}^+)\delta(\psi_{y,t} - \psi_{y,t}{}^-)$. As opposed to a single parameter description for the isotropic case, anisotropy involves two parameters $(\beta \neq 0)$ implying non-zero transition probabilities. This added complication renders an asymmetric $(\psi_{x,t} \leftrightarrow \psi_{y,t})$ structural form resulting in

$$\alpha_t = \frac{F(\psi_{x,t}{}^\pm, \psi_{y,t}{}^\pm)}{2N(\psi_{x,t}{}^\pm, \psi_{y,t}{}^\pm)}, \quad \beta_t = \frac{H(\psi_{x,t}{}^\pm, \psi_{y,t}{}^\pm)}{2N(\psi_{x,t}{}^\pm, \psi_{y,t}{}^\pm)} \qquad (9)$$

where F, H and N are functions of $\psi_i{}^\pm(t)$. As in the $t \to \infty$ case, we find that $s_{x,t} = -\psi_{x,t}{}^+\psi_{x,t}{}^-$ and $s_{y,t} = -\psi_{y,t}{}^+\psi_{y,t}{}^-$. These relations leave us with only two independent variables determining the non-stationary state dynamics as follows
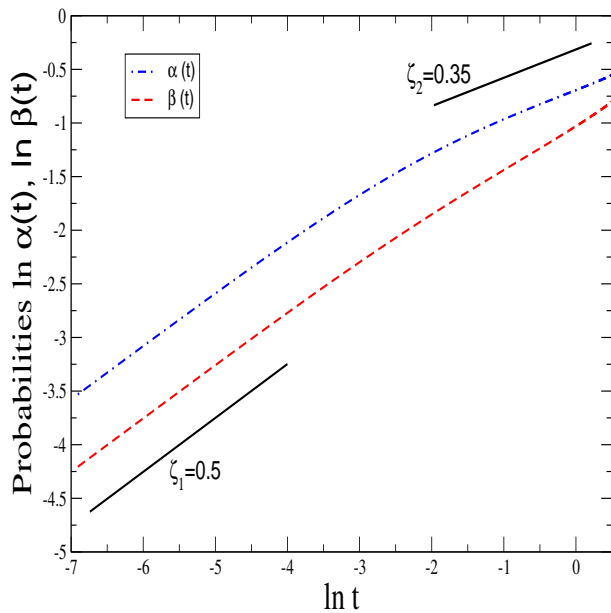
FIG. 1: (Color online) Variations of α_t and β_t with time t on a log-log scale. The former shows a cross-over behavior while the latter depicts a steady scaling. The result is typical of the *strong coupling regime*.
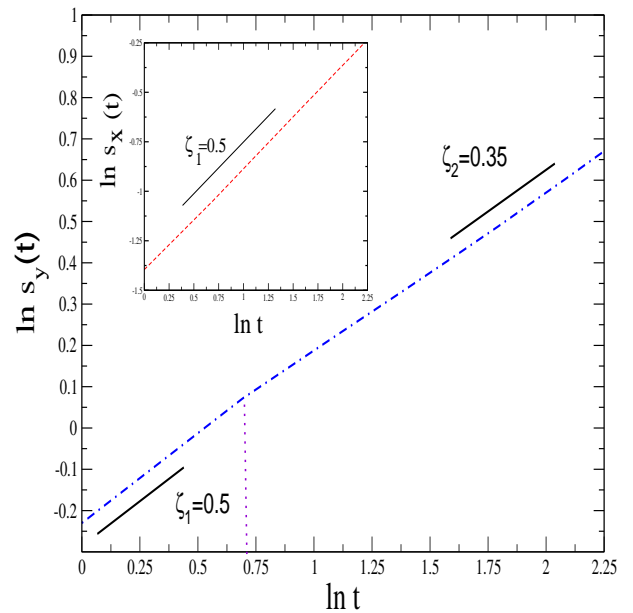

FIG. 2: (Color online) Variations of $\ln s_x(t)$ and $\ln s_y(t)$ with $\ln t$ using simulation data of eqns (3) and (4) for $\xi = 10$, $D = 0.5$. The $\ln s_y(t)$ graph shows a cross-cross-over scaling akin to the $\ln \alpha_t$ graph in Fig. 1 thereby reconfirming our theoretical result. The inset plots $\ln s_x(t)$ against $\ln t$ for a steady scaling exponent 0.5.

$$\frac{d\alpha_t}{dt} = \exp\Big[-\frac{V({\psi_{x,t}}^-,{\psi_{y,t}}^-) - V({\psi_{x,t}}^+,{\psi_{y,t}}^+)}{D}\Big]$$
(10a)

$$\frac{d\beta_t}{dt} = \exp\Big[-\frac{V({\psi_{x,t}}^-,{\psi_{y,t}}^+) - V({\psi_{x,t}}^+,{\psi_{y,t}}^-)}{D}\Big]$$
(10b)

Equations (10a) and (10b) are of critical importance in that they define the probabilities of transition between two "pure" minima and two "mixed" states respectively.

We numerically solved the coupled set of equations (10a, 10b) for the parameter values $\xi = 10$, $D = 0.5$ (the conclusions remain unchanged over a considerable range of parameters) and arrived at a remarkable result as in Fig. 1: the $\ln \alpha_t$ vs t plot shows a cross-over from the temporal exponent $\zeta_1 \sim 0.5$ to $\zeta_2 \sim 1/3$ whereas the $\ln \beta_t$ versus t graph shows a steady scaling $\zeta_1 \sim 0.5$. Such a cross-over is unknown in the *weak coupling regime* ($\zeta = 0.5$) [2, 26].

In an attempt to understand the true implication of this scaling structure as well as to check the consistency of our results, we numerically integrated eqns (3) up to 10^6 time steps for $\xi > D$ and found this sensational scaling tenable in the strong coupling regime over *six orders of magnitudes* (Fig.2). The plot clearly confirms the cross-over scaling behavior obtained from the *non-perturbative* theory (Fig. 1). For the special case of $g_x = g_y$ ($\xi = 1$), we recover the Marsili-Bray result [8]: $s_t = (lnt)^{1/3}$.

The spatial correlation function, on the other hand, shows an exponential decay as evident from Fig. 3 that confirms the *absence of any spatial scaling* in the anisotropic SC regime. These two figures, Fig. 2 and Fig. 3 combined, reveal a startling fact about the strong coupling regime – although the SC regime exhibits a most interesting temporal scaling behavior, there is no spatial scaling. In other words, there is an absence of an underlying renormalization group fixed point in the SC regime so far as spatio-temporal scaling is concerned.

The overall absence of a "true scaling" in the strong coupling KPZ model is a singularly striking result and contradicts previous predictions [8, 9]. We attribute such a difference in conclusion to the non-exactness that accompanies a perturbative evaluation in a non-perturbative regime [9] as opposed to our case where the entire analysis relies on the stability behaviors of a non-perturbative potential function around the points of stability, generally the attractors. The spatio-temporal spectrum in the SC regime further allows us to make a prediction concerning the upper critical dimension d_c below which WC scaling holds (and above which $\chi = 0$). A lack of spatial scaling, as evident from Fig. 3, simply renders this number as infinite and can be directly related to the fact that both $\alpha(t), \beta(t)$ are always greater than zero. It might be noted that for an isotropic system where $\beta = 0$, the argument still holds, giving $d_c = \infty$. The result agrees with earlier predictions [3, 9, 14] although, admittedly, disagrees with many others [11, 19–
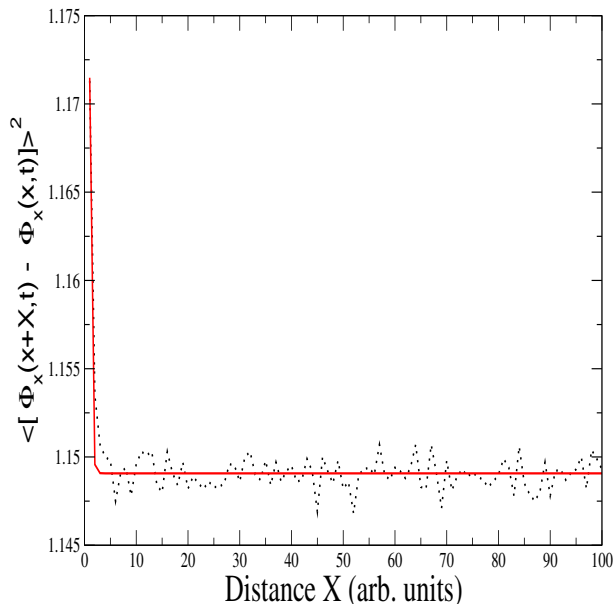
FIG. 3: (Color online) Spatial correlation function as obtained from exact simulation data (dotted line) for $\xi = 10$, $D = 0.5$. The solid line is an exponential fit. The results show a typical exponential decay predicting a lack of spatial scaling in the *strong coupling regime*.

22]. Our numerical studies of other non-linear models in the SC regime reconfirm the lack of spatial scaling and an asymmetry dependent transition in temporal scaling. One must be a bit cautious about the approximations though. A nearest-neighbour interaction in mapping the continuum model to its discrete equivalent and overall dynamic scaling, our basic assumptions, could be quantitatively non-trivial (amounting to modified values of exponents) although the qualitative outcomes should still remain unaffected. A problem that is worth pursuing is a study of the behaviors of similar strongly-coupled models in the presence of a multiplicative noise, especially for a quenched system (SC equivalent of [27]).

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